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Bits, matrices and $1/N$

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Abstract: We propose a simple string bit formalism for interacting strings in a plane wave background, in terms of supersymmetric quantum mechanics with a symmetric product target-space. We construct the light-cone supersymmetry generators and hamiltonian at finite string coupling. We find a precise match between string amplitudes and the non-planar corrections to the correlation functions of BMN operators computed from gauge theory, and conjecture that this correspondence extends to all orders in perturbation theory. We also give a simple RG explanation for why the effective string coupling is $g_2 = J^2/N$ instead of $g_s = g_{ym}^2$.

Keywords: 1/N Expansion, Penrose limit and pp-wave background.

Contents

1. Introduction

Recently a new framework to describe type IIB string dynamics from $\mathcal{N}=4$ supersymmetric Yang-Mills theory has been proposed in [1]. It was argued that for states with large R-charge J, the gauge theory in effect amounts to a (discrete) light cone quantization of strings moving in the background of a pp-wave geometry [2]

$$ds^2 = -4dx^+dx^- - \mu^2(\vec{r}^2+\vec{y}^2)(dx^+)^2 + d\vec{y}^2 + d\vec{r}^2\,, \qquad F_{+1234} = F_{+5678} = \frac{\mu}{4\pi^3 g_s \alpha'^2}\,. \tag{1.1}$$

The specific dictionary proposed in [1] involves a one-to-one identification of a class of single trace operators in the gauge theory with single string states in this background. At the non-interacting string level, at leading order in the large N expansion, supporting evidence was found via the computation of the anomalous scale dimensions of the gauge theory operators, confirming the identification

$$P_- = \Delta - J\,. \tag{1.2}$$

Subsequent work has extended this correspondence to low orders in string perturbation theory.

In this note, we will propose a simple effective description of the string dynamics in the pp-wave background, in terms of a string bit language [4] inspired by the formalism of [1] as well as by matrix string theory [5, 6]. This reduced description is based on a simple supersymmetric quantum mechanical model with a symmetric product target space. We will present evidence that this effective description exactly reproduces the *complete* perturbation expansion of the $\mathcal{N}=4$ SYM theory in the BMN limit. In particular, we will find a precise match between the amplitudes of the string bit model and the three-point proposed in [10] and one-loop amplitudes computed in [9] and [10]. The philosophy of our approach, however, is to compare the gauge and string theory at a more microscopic level, by matching both systems directly via their light-cone hamiltonian evolution.

The light-cone gauge worldsheet lagrangian of a string propagating in the background (1.1) takes the quadratic form

$$\mathcal{L} = \frac{1}{2}\left(\partial_+ x_i \partial_- x_i - \mathrm{m}^2 x_i^2\right) + \mathrm{i}\left(\theta_a \partial_+ \theta_a + \tilde{\theta}_a \partial_- \tilde{\theta}_a - 2\mathrm{m}\tilde{\theta}_a \Pi_{ab} \theta_b\right), \tag{1.3}$$

with $\mathrm{m} = \mu\, p_+$ and $\Pi = \gamma_{1234}$. The quantization and symmetries of this lagrangian have been studied in detail in [3]. In accordance with the symmetries of the pp-wave background, it exhibits a maximally extended sypersymmetry (for all details, see [3]).

In the following, our aim is to construct an *interacting* version of this string theory, with a discretized world-sheet and with the same space-time symmetries. The only generators that are expected to receive corrections at finite string coupling are the light-cone generators Q^- and P^-. Hence, the part of the supersymmetry algebra that will be most important to us, is[1]

$$\{Q^{\dot a}, Q^{\dot b}\} + \{\widetilde{Q}^{\dot a}, \widetilde{Q}^{\dot b}\} = \delta^{\dot a \dot b} H + \mathrm{m} J^{\dot a \dot b}, \qquad [H, Q_{\dot a}] = 0\,, \tag{1.4}$$

where $H \equiv P^-$ and $J^{\dot a \dot b}$ is a suitable contraction of gamma matrices with the $SO(4)\times SO(4)$ Lorentz generators J^{ij}, see [3].

A convenient formalism for describing the full multi-string Hilbert space, is to use a single orbifold field theory on the symmetric product target space

$$\mathrm{Sym}^J \mathcal{M} = \frac{\mathcal{M}^J}{S_J} \tag{1.5}$$

with S_J the permutation group and J the total DLCQ momentum. In the case of IIA string theory in flat space, this formalism naturally arises as the IR limit of matrix string theory [5, 6].[2] Moreover, it allows for a remarkably compact description of the string splitting and joining, via the interaction vertex [6]

$$g_s V_{\mathrm{int}} = \frac{g_s}{J^2} \sum_{n<m} \int d\sigma \left(\tau^i \tilde{\tau}^j \otimes \Sigma^i \widetilde{\Sigma}^j\right)_{nm}, \tag{1.6}$$

where each term on the right-hand side represents a suitable twist-field that implements a simple permutation (nm), interchanging the n-th and m-th copy of the target space. This description of matrix string interactions will be useful to keep in mind in what follows.

2. Bits

Motivated by the BMN formalism, we will now set up the string bit language, by introducing J copies of supersymmetric phase space coordinates $\{p_n^i, x_n^i, \theta_n^a, \tilde{\theta}_n^a\}$, with $n = 1, \ldots, J$, satisfying canonical commutation relations

$$[p_n^i, x_m^j] = \delta^{ij}\delta_{mn}\,, \qquad \left\{\theta_n^a, \theta_m^b\right\} = \delta^{ab}\delta_{mn}\,, \qquad \left\{\tilde{\theta}_n^a, \tilde{\theta}_m^b\right\} = \delta^{ab}\delta_{mn}\,. \tag{2.1}$$

Following the example of matrix string theory, we consider these J copies as obtained via the quantization of the J-th symmetric product $\mathrm{Sym}_J\mathcal{M}$ of the plane wave target space $\mathcal{M}$. In other words, we divide out the symmetric group S_J, acting via permutation of on the labels n, thus defining a quantum mechanical orbifold model.

[1] Here, relative to the notation in [3], $Q = Q^{-1}$ and $\widetilde{Q} = Q^{-2}$.

[2] For some recent work on matrix (string) theory in pp-wave backgrounds, see [12]–[14].

In the context of 2-d field theory, it is well known that the Hilbert space of such an orbifold field theory decomposes as a direct sum over twisted sectors. Although the occurrence of such twisted sectors is often thought of as a purely "stringy" phenomenon, they do in fact also arise in point-particle quantum mechanics. For a clear discussion of this in the context of cohomology of symmetric orbifold spaces, see [7, pp. 56–57.]. Following the prescription outlined in [7], we construct our Hilbert space as a direct sum of "twisted sectors"

$$\mathcal{H} = \bigoplus_{\gamma} \mathcal{H}_\gamma \tag{2.2}$$

labeled by conjugacy classes γ of the symmetric group S_J. Each twisted sector $\mathcal{H}_\gamma$ can be thought of as made up from states localized at the fixed point set of γ.[3] This fixed point set is mapped onto itself by the stabilizer subgroup N_γ of permutations σ that commute with γ. Correspondingly, in $\mathcal{H}_\gamma$, we can act with arbitrary operators $\mathcal{O}(p, x, \theta)$ that are left *invariant* under the action of N_γ:

$$\{\, p_n^i, x_n^i, \theta_n^a\} \to \left\{p^i_{\sigma(n)}, x^i_{\sigma(n)}, \theta^a_{\sigma(n)}\right\}\,, \qquad \sigma \in N_\gamma\,. \tag{2.3}$$

By copying (with only slight modification) the discussion in [6] (on pp 4-5), it is readily verified that the resulting Hilbert space takes the form of a sum over multi-string Hilbert spaces, each string with a discretized worldsheet consisting of J_ℓ bits with $\sum_\ell J_\ell = J$. As in matrix string theory, we interpret J_ℓ as the discrete light-cone momentum of the string. The above invariance under the stabilizer subgroup imposes the constraint

$$(\,U_\ell)^{J_\ell} = 1 \qquad U_\ell = e^{2\pi i (L_0^{(\ell)} - \widetilde{L}_0^{(\ell)})/J_\ell} \tag{2.4}$$

on each string, with U_ℓ the operator that translates the string bit x_n by one unit. Here the adjacency of two sites is specified by the twisted boundary condition γ. So in particular the "overall" translation operator $U = \bigotimes_\ell U_\ell$ is defined to act via

$$U X_n U = X_{\gamma(n)} \tag{2.5}$$

with $X_n = \{p_n^i, x_n^i, \theta_n^a\}$. It translates the bits only within each individual string, i.e. it does not translate any bit from one string to the next. Finally, we notice that each multi-string state is required to be (anti-)symmetric (according to statistics) under interchange of strings with equal discrete light-cone momentum J_ℓ.

The light-cone supersymmetry generators and hamiltonian of the free string theory read

$$Q_0 = Q_{\dot a}^{(0)} + \lambda\, Q_{\dot a}^{(1)}\,, \qquad H_0 = H^{(0)} + \lambda\, H^{(1)} + \lambda^2 H^{(2)} \tag{2.6}$$

[3]Here and in the following, we use the notation γ both for the conjucacy class as for a specific representative in this class.

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with

$$
\begin{aligned}
Q^{(0)}_{\dot a} &= \textstyle\sum_n \left(p^i_n \gamma_i\, \theta_n - x^i_n (\gamma_i \Pi)\tilde\theta_n\right) , & Q^{(1)}_{\dot a} &= \textstyle\sum_n \left(x^i_{\gamma(n)} - x^i_n\right) \gamma_i\, \theta_n \\
H^{(0)} &= \sum_n \left(\frac{1}{2}\left(p^2_{i,n} + x^2_{i,n}\right) + 2\mathrm{i}\, \tilde\theta_n \Pi \theta_n\right) , & & \\
H^{(1)} &= -\textstyle\sum_n \mathrm{i}\left(\theta_n \theta_{\gamma(n)} - \tilde\theta_n \tilde\theta_{\gamma(n)}\right) , & H^{(2)} &= \sum_n \frac{1}{2}\left(x^i_{\gamma(n)} - x^i_n\right)^2 .
\end{aligned}
\tag{2.7}
$$

These expressions are a straightforward discretization of the expressions in [3]. Here λ is a parameter that controls the size of the string bits relative to the mass scale m in the light-cone worldsheet lagrangian (1.3).

Via [1] and [11], there now exists convincing evidence that the above free light-cone hamiltonian exactly summarizes the propagation of the BMN operators in the leading order large N limit, via the identification (1.2) and of λ with the 't Hooft coupling

$$\lambda^2 = g^2_{ym} N . \tag{2.8}$$

Our next goal is to extend this correspondence to include string splitting and joining.

3. Interactions

We now add interaction terms to the light-cone generators, as follows

$$Q^{\dot a} = Q^{\dot a}_0 + g_s S^{\dot a}_1 , \qquad H = H_0 + g_2 V_1 + g_2^2 V_2 , \tag{3.1}$$

with g_2 the (effective) string coupling. Imposing the light-cone supersymmetry algebra (1.4) produces the relations

$$\left\{Q^{\dot a}_0, S^{\dot b}_1\right\} + \left\{\widetilde Q^{\dot a}_0, \widetilde S^{\dot b}_1\right\} + (\dot a \leftrightarrow \dot b) \;=\; \delta^{\dot a \dot b}\, V_1 \tag{3.2}$$

$$\left[H_0, S^{\dot a}_1\right] + \left[V_1, Q^{\dot a}_0\right] \;=\; 0 \tag{3.3}$$

$$\left\{S^{\dot a}_1, S^{\dot b}_1\right\} + \left\{\widetilde S^{\dot a}_1, \widetilde S^{\dot b}_1\right\} \;=\; \delta^{\dot a \dot b}\, V_2 \tag{3.4}$$

$$\left[V_2, Q^{\dot a}_0\right] + \left[V_1, S^{\dot a}_1\right] \;=\; 0 . \tag{3.5}$$

We wish to solve these relations via suitable interaction terms S_1 and V_1 that induce the splitting and joining of strings. It will be a non-trivial result that the above algebra can be satisfied, without the need of introducing any higher order interaction terms than V_2.

Following the matrix string theory example, it is natural to look for operators analogous to the twist-field interaction (1.6). Consider operators Σ_{mn} that implement a simple transposition of two string bits via

$$\Sigma_{nm} X_m = X_n\, \Sigma_{nm} , \qquad \Sigma_{nm} X_k = X_k\, \Sigma_{nm} \qquad k \neq m, n \tag{3.6}$$

with $X_n = \{p^i_n, x^i_n, \theta^a_n\}$. Clearly, by acting with Σ_{mn} on a given multi-string sector, we get a different multi-string sector via

$$\Sigma_{mn} : \quad \mathcal{H}_\gamma \to \mathcal{H}_{\tilde\gamma} , \qquad \text{with } \ \tilde\gamma = \gamma \circ (mn) . \tag{3.7}$$

Depending on whether the two sites n and m in the sector γ correspond to one single string (say of length J_0) or two separate ones (say of length J_1 and J_2), the new sector $\tilde{\gamma}$ is obtained by either splitting the single string in two pieces of length $(m-n)$ and (J_0-m+n), or by joining the two strings to one of length J_1+J_2.

Let us now introduce the S_J invariant operator

$$\Sigma = \frac{1}{J^2} \sum_{n<m} \Sigma_{nm} \tag{3.8}$$

and define the interaction terms via

$$S_1^{\dot{a}} = \left[Q_0^{\dot{a}}, \Sigma\right] , \qquad V_1 = [H_0, \Sigma] . \tag{3.9}$$

This form of the interaction term has several motivations. First, it manifestly solves the equations (3.2) and (3.3) imposed by the space-time supersymmetry algebra. Secondly, by the preceding discussion, it represents a simple splitting and joining interaction. Finally, the matrix string theory vertex (1.6), light-cone string field theory [8] and the $\mathcal{N}=4$ gauge theory calculations [10], all suggest that the interaction vertex should contain, besides the simple splitting and joining operator, a prefactor quadratic in the string oscillators. Moreover, we expect that the interaction term should vanish when $\lambda = 0$, i.e. in the free SYM theory. (We will further motivate this requirement in the next subsection.) Notice that, since the leading order generators $Q^{(0)}$ and $H^{(0)}$ are permutation invariant, both interaction terms are indeed (at least) of order λ.

It is straightforward, via (3.6) and the definition of the light-cone generators, to explicitly compute S_1 and V_1. For S_1 we find

$$S_1 = \frac{1}{J^2} \sum_{m,n} \Sigma_{mn} \left(\theta_m \gamma^i \left(x^i_{\gamma(m)} - x^i_{\gamma(n)}\right) + \theta_{\gamma^{-1}(m)} \gamma^i \left(x^i_n - x^i_m\right) - \delta_{n\gamma(m)} \left(\theta_m \gamma^i x^i_n - \theta_n \gamma^i x^i_m\right)\right) . \tag{3.10}$$

From this explicit form, it is directly verified that the S_1 terms in fact satisfy a commutation relation of the form (3.4), which can thus be taken as the definition of the second order interaction term V_2. The relation (3.5) is then automatically satisfied.

Thus the interacting string-bit theory has all the required symmetries, as well as all required dynamic properties (at least all obvious ones). However, since the time of the first version of this paper, several further developments led to a more precise understanding of both the gauge theory and string theory side of the story, which made clear that the original proposal as formulated above needs some modification.

Initially, the interaction vertex as proposed in eq. (3.9) coincided not only with the reported values of the gauge theory amplitudes [9, 10] but also with the three-point function in string field field theory, as reported in the original version of [19]. Soon after, however, more careful study of the gauge theory amplitudes [20, 21], (in part prompted by our suggestion that operator mixing must in fact be taken into account) resulted in a new and different answer for the three-point function, as well as for the shift in the conformal dimension of the two-impurity states [20, 21]. Simultaneously, it was pointed out [22] that the original string theory calculation of [19] contained a subtle sign error, and needed correction as well.[4]

In [25], the following refinement of the bit string vertex (3.9) was proposed

$$S_1^{\dot{a}} = \left[\widehat{Q}_0^{\dot{a}}, \Sigma\right] , \qquad \widehat{Q}_0 = Q_0^< - Q_0^> , \tag{3.11}$$

where $Q_0 = Q_0^< + Q_0^>$ is the free supercharge of the bit string theory and the superscripts indicate the projection onto the term with fermionic creation ($<$) or annihilation ($>$) operators only. These charges generate the interacting superalgebra of string theory in the plane wave background, modulo higher order terms in the fermions.

The supercharge (3.11) truncates at linear order in g_2. This truncation is expected from the BMN correspondence: matrix elements of the supercharges $Q = \mathrm{Tr}\,\theta[Z, \phi]$ in the gauge theory can lead (for connected diagrams, where Q has at least one contraction with either the "in" or "out" BMN state) to at most one single string splitting or joining [25]. For now, however, one may view (3.11) as a new starting point of the string bit model. From (3.11) we deduce that the interacting hamiltonian truncates at order g_2^2:

$$H = H_0 + g_2 H_1 + g_2^2 H_2 \tag{3.12}$$

with[5]

$$H_0 = \{Q_0, Q_0\} , \qquad H_1 = \left\{Q_0, \left[\widehat{Q}_0, \Sigma\right]\right\} , \qquad H_2 = \frac{1}{4}\left\{\left[\widehat{Q}_0, \Sigma\right], \left[\widehat{Q}_0, \Sigma\right]\right\} . \tag{3.13}$$

We will now establish its equivalence with the perturbative gauge theory, by showing that, at least for the special class of two-impurity states, it leads to the same order g_2^2 mass renormalization (shift in conformal dimension) as computed in [20, 21].

4. Matrices

The interacting string bit theory is constructed to correspond with the non-planar gauge theory amplitudes in the BMN limit, via the identification of the effective string coupling g_2 with

$$g_2 = \frac{J^2}{N} , \tag{4.1}$$

and of the light-cone energy with scale dimension $\Delta - J$. In view of this correspondence, it seems appropriate to view the gauge theory amplitudes as obtained via radial evolution starting from an initial state defined at some given point, say at $r = 0$, which then evolves via the dilation operator to a final state defined at $r = \infty$. Preferably, from a given initial operator at $r = 0$, one would then like to be able to keep track of how the index structure

[4] An alternative construction of the string interaction vertex, which in fact bears close resemblance to the vertex (3.9), was proposed in [23, 24].

[5] The notation below is somewhat symbolic: the left-hand side in each equation is equal to the projection onto the $\delta^{\dot{a}\dot{b}}$ component of the anti-commutator on the right-hand side.

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of the state evolves with r, so that, via an appropriate dictionary, we can identify the number of strings propagating at this instant and make a precise comparison with the corresponding light-cone string diagram.

A technical fact, that seems to be at odds with this philosophy, is that the two-point functions of single and multiple trace operators in the gauge theory in fact have a non-trivial $1/N$ expansion even at $g_{ym} = 0$, [9, 16] and [10]. This would seem to indicate that strings can split and join even without any non-trivial SYM interaction taking place, thus preventing a precise determination of the number of strings at given light-cone time r. Our point of view, however, is that the $1/N$ corrections that are present in the free SYM theory, do *not* correspond to proper string interactions, but need to be absorbed into a redefinition of the precise operator dictionary (using techniques similar to [15]). This redefinition is possible, because string S-matrix elements between in- and out-states, via the above prescription, in fact correspond to *two-point* functions in SYM theory of corresponding multi-trace operators placed at $r = 0$ and $r = \infty$, rather than multi-point functions. In particular, the three-point function is an S-matrix element between an one-string and a two-string state (see below), and *not* equal to a three-point function of the SYM theory.[6]

Modulo this subtlety, single trace operators in first order in g_2 correspond to single strings. The cyclicity of the trace then naturally implements the $L_0 - \widetilde{L}_0$ constraint (2.4), while via permutations of the operators in a single or multi-trace expression, we can (similarly as discussed above) induce string splitting or joining. Combined with the original analysis of [1], it seems natural to look for a combinatoric language by which we can characterize the SYM interactions as a composition of a small number of elementary operations, such as nearest neighbor hopping or simple permutations. The general dictionary relies on a one-to-one correspondence between interaction terms in the SYM and string bit hamiltonian.

Let us now make this correspondence explicit. Let us denote by $|\psi_n\rangle$ the basis defined by single and multiple trace operators in $\mathcal{N} = 4$ gauge theory in the BMN limit. This basis is orthogonal in the free gauge theory at $g_2 = 0$. However, as mentioned above, at non-zero g_2 the inner product becomes non-diagonal: it has been shown in [25] that this inner product is conveniently expressed in string bit language as follows[7]

$$\langle\psi_m|\psi_n\rangle_{g_2} = \left(e^{g_2\Sigma}\right)_{nm} . \tag{4.2}$$

The identification of (4.2) with the inner product of the free gauge theory was motivated in [25] and explicitly verified for string ground states to all order in g_2 and for two-impurity

[6]This point of view was also taken in [10].

[7]To define Σ in the gauge theory language, consider a long BMN string in its ground state. We can write the corresponding operator as $\mathcal{O}_J(\gamma) = \mathrm{Tr}(Z^J) = \sum_{\substack{i_1\ldots i_J \\ \bar{i}_1\ldots\bar{i}_j}} Z_{i_1\bar{i}_1}\, Z_{i_2\bar{i}_2} \cdots Z_{i_J\bar{i}_J}\ \delta^{\bar{i}_1 i_{\gamma(1)}}\,\delta^{\bar{i}_2 i_{\gamma(2)}}\ldots\delta^{\bar{i}_J i_{\gamma(J)}}$ with $\gamma = (12\ldots J)$ the cyclic permutation of J elements. The action of $\Sigma_{J_1 J}$, which implements the simple permutation $(J_1 J)$, is now defined as $\Sigma_{J_1 J}\,\mathcal{O}_J(\gamma) = \mathcal{O}_J(\gamma\circ(J_1 J))$. Since $\gamma\circ(J_1 J) = (1\ldots J_1 - 1\,J)(J_1\ldots J-1)$ we have that $\Sigma_{J_1 J}\,\mathcal{O}_J(\gamma) = \mathrm{Tr}(Z^{J_1})\,\mathrm{Tr}(Z^{J-J_1})$, showing that the simple permutation $\Sigma_{J_1 J}$ indeed induces a single splitting of a single trace into a double trace operator. It is easy to generalize this result to other operators, to show that Σ can either split a string or join two strings.

states to order g_2^2. States with different number of traces are therefore no longer orthogonal relative to (4.2). In the string theory basis, denoted by $|\tilde{\psi}_n\rangle$, the inner product should be diagonal for all g_2. The simplest basis transformation that achieves this goal is:

$$|\widetilde{\psi}_n\rangle = \left(e^{-g_2\Sigma/2}\right)_{nm} |\psi_m\rangle\,. \tag{4.3}$$

In the following, we will use this basis transformation in our comparison between the string bit and gauge theory amplitudes for the special case of the two-impurity BMN states investigated in [9, 10, 20, 21, 27]. We denote by $|1,p\rangle$ the normalized state corresponding to the single trace operator $\sum_l e^{2\pi i p l/J}\,\mathrm{Tr}(\phi Z^l \psi Z^{J-l})$, while $|2,k,y\rangle$ and $|2,y\rangle$ will denote the normalized states corresponding to the double trace operators $\sum_l e^{2\pi i k l/J_1}\,\mathrm{Tr}(\phi Z^l \psi Z^{J_1-l}) \times \mathrm{Tr}(Z^{J-J_1})$ and $\mathrm{Tr}(\phi Z^{J_1})\,\mathrm{Tr}(\psi Z^{J-J_1})$ respectively, where $y = J_1/J$. The action of Σ on $|1,p\rangle$ can be straightforwardly calculated

$$\Sigma|1,p\rangle = \sum_{k,y} C_{pky}\,|2,k,y\rangle + \sum_{y} C_{py}|2,y\rangle\,, \tag{4.4}$$

with

$$C_{pky} = \sqrt{\frac{1-y}{Jy}}\,\frac{\sin^2(\pi p y)}{\pi^2(p-k/y)^2}, \qquad C_{py} = -\frac{\sin^2(\pi p y)}{\sqrt{J}\pi^2 p^2}\,. \tag{4.5}$$

Via (4.3) we now introduce the corresponding two-impurity states in the string basis, which we will denote by $|\widetilde{1},p,y\rangle$, $|\widetilde{2},k,y\rangle$ and $|\widetilde{2},y\rangle$, respectively. By construction, these form an orthonormal basis at finite g_2.

From the explicit form of the string bit hamiltonian, it is straightforward to compute the matrix elements between the various two-impurity states. Using that $Q_0^>$ annihilates bosonic states, we find that

$$\langle\widetilde{\psi}_2|H_1|\widetilde{\psi}_1\rangle = \frac{1}{2}\langle\widetilde{\psi}_2|(H_0\Sigma+\Sigma H_0)|\widetilde{\psi}_1\rangle - 2\langle\widetilde{\psi}_2|Q_0^>\,\Sigma\,Q_0^<\,|\widetilde{\psi}_1\rangle\,. \tag{4.6}$$

Both matrix elements on the right-hand side have been computed in [25], with the result

$$\langle\widetilde{2},k,y|(H_0\Sigma+\Sigma H_0)|\widetilde{1},p\rangle \;=\; \lambda'(p^2+k^2/y^2)\,C_{pky} \tag{4.7}$$

$$\langle\widetilde{2},k,y|\,Q_0^>\Sigma\,Q_0^<\,|\widetilde{1},p\rangle \;=\; \frac{\lambda'}{2}\,(pk/y)\,C_{pky}\,, \tag{4.8}$$

where C_{pky} are the three point functions defined in eqn (4.5). Inserting the explicit expressions, we find

$$\langle\widetilde{2},k,y|H_1|\widetilde{1},p\rangle = \frac{\lambda'}{2}\sqrt{\frac{1-y}{Jy}}\frac{\sin^2(\pi p y)}{\pi^2}, \qquad \langle\widetilde{2},y|H_1|\widetilde{1},p\rangle = -\frac{\lambda'}{2}\frac{1}{\sqrt{J}}\frac{\sin^2(\pi p y)}{\pi^2}\,. \tag{4.9}$$

In a similar way one can obtain the order g_2^2 matrix elements between single string states. The computation is described in [25] (see equation (71)), with the result

$$\begin{aligned} g_2^2\langle\widetilde{1},q|H_2|\widetilde{1},p\rangle \;&=\; \frac{g_2^2}{4}\langle\widetilde{1},q|\;[Q_0^>,\Sigma]\;[\Sigma,Q_0^<\,]\;|\widetilde{1},p\,\rangle \\ &=\; \frac{g_2^2\lambda'}{4}\sum_i\left(\frac{k^2}{y^2}-\frac{1}{2}(p^2+q^2)\right)C_{p\,i}C_{qi} \;\equiv\; \frac{1}{4}\frac{g_2^2\lambda'}{4\pi^2}\,B_{pq}\,. \end{aligned} \tag{4.10}$$

Here the sum runs over intermediate 2-string states i of both types: $|\widetilde{2},k,y\rangle$ and $|\widetilde{2},y\rangle$ (and includes an integral over y). The explicit form of B_{pq} is as given in [20, 21].

To obtain the order g_2^2 mass renormalization of the state $|\widetilde{1},p\rangle$, we should add the matrix element (4.10) to the iterated H_1 interaction:

$$
\begin{aligned}
\sum_i \frac{|\langle \widetilde{1},p\,|\,H_1\,|\,i\,\rangle|^2}{E_p - E_i} &= \frac{1}{4} g_2^2 \lambda' \sum_i \frac{(p-k/y)^4 C_{pi}^2}{p^2 - k^2/y^2} \\
&= \frac{1}{4} g_2^2 \lambda' \sum_i \left(p^2 - \frac{k^2}{y^2}\right) C_{pi} C_{-pi} \\
&= -\frac{1}{4} \frac{g_2^2 \lambda'}{4\pi^2} B_{p,-p}\,. \qquad (4.11)
\end{aligned}
$$

Putting everything together, we find that the order g_2^2 contribution to the eigenvalue is

$$
\frac{1}{4} \frac{g_2^2 \lambda'}{4\pi^2} (B_{pp} - B_{p,-p}) = \frac{g_2^2 \lambda'}{4\pi^2} \left(\frac{1}{12} + \frac{35}{32\pi^2 p^2} \right). \qquad (4.12)
$$

This agrees precisely with the shift in anomalous dimension of the conformal eigen-operators, as reported in [20, 21].

5. J^2/N versus g_s

A remaining puzzle is to explain why the effective string coupling needs to be identified with $g_{{}_2} = J^2/N$, rather than with the fundamental string coupling

$$
g_s = \frac{g_2 \lambda^2}{J^2} \qquad (5.1)
$$

predicted by the AdS/CFT dictionary. An intuitive explanation is that strings in the plane wave background are in effect confined to a finite transverse volume, which amounts to an effective dimensional reduction to two dimensions and a corresponding rescaling of the Newton constant [10]. It is important, however, to understand this renormalization also directly from the worldsheet perspective.[8]

The physical meaning of the dimensionless parameter λ is that it represents the (inverse) size of the string bit relative to the worldsheet mass-scale m created by the plane wave background. Calling the string bit size ℓ, we have

$$
\mathrm{m}\,\ell = \frac{1}{\lambda}\,. \qquad (5.2)
$$

The SYM perturbative regime therefore corresponds to the limit where the string bit size is much larger than the length scale set by m. There is no reason, however, certainly not within the string bit model, to assume that λ is small.

In the limit of large λ and large J, there will be an intermediate regime of worldsheet scales L at which the string bits look infinitesimal, while the mass perturbation m is still small. In this regime, the worldsheet dynamics should look close to that of strings in flat

[8]Ideas similar to the following discussion were put forward in [12].

space. In particular, the string interactions should have an accurate effective description by means of the matrix string interaction vertex (1.6). In addition, the effective string coupling should be equal to g_s rather than g_2.

Let us compare the two effective interaction vertices (1.6) and (3.9). Both are a product of a pure twist-field (splitting/joining) interaction and a quadratic expression in the string oscillators. It indeed looks like it should be possible to continuously connect the two operators by means of a suitable renormalization group trajectory, connecting the small λ regime (3.9) with the large λ regime (1.6). This problem is presently under study in [17].

On general grounds, however, one can already deduce how the effective couplings should be related in the two regimes. Namely, a pure twist field interaction $V_{\rm twist} = \sigma\tilde{\sigma}\Sigma\tilde{\Sigma}$ in the massless continuum worldsheet theory has scale dimension 2. In the massive theory, however, this conformal dimension will depend non-trivially on the ratio (5.2): it interpolates from 2 for λ large, to 0 for small λ, [17]. Since the typical length scale of the worldsheet dynamics is of order $J\ell$, one expect that under the RG flow the twist field gets renormalized with a factor of $({\rm m}J\ell)^2$. That is

$$V_{\rm twist} \leftrightarrow \Sigma\frac{J^2}{\lambda^2} \tag{5.3}$$

with Σ the string bit version (3.8) of the twist field. This explains the renormalization (5.1) of the effective string coupling.

6. Conclusion and outlook

We have presented a simple string bit system that describes interacting strings in a plane wave background. The model is complete in the sense that the space-time symmetry algebra closes at finite string coupling, without the need for higher order terms other than that we have described here. The BMN conjecture is that the free string bit hamiltonian correctly summarizes all planar interactions of $\mathcal{N} = 4$ gauge theory in the BMN limit; via our model, one is now free to extend this conjecture to arbitrary orders in the string loop expansion. While in principle it is possible that extra terms may (need to) be added to H, these are highly constrained by the requirement that they preserve the space-time supersymmetry algebra, as well as the existence of a continuum limit.

The vertex (3.11) has been shown to reproduce all known gauge theory amplitudes [20, 21]. In addition, a basis transformation was proposed that relates the gauge theory basis of single and multi-trace operators with the string theory basis of single and multi-string states. This proposed dictionary has verified to linear order in g_2 in [26].[9]

Our general approach of identifying string scattering amplitudes with two-point functions of multi-trace operators, rather than multi-point functions, has been supported by several later results. In particular, in [30] an effective quantum mechanical model for capturing the gauge theory interactions was constructed, which was shown in [31] to be in accord with the string bit vertex (3.11).

[9]This dictionary was independently proposed in [27] and [28], and verified to second order in [29].

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What is the regime of validity of the model? From the interaction hamiltonian, we find that string interactions remain weak as long as

$$\frac{g_2 \lambda}{J} = \frac{g_{ym} J}{\sqrt{N}}\,, \qquad \text{and} \qquad \frac{g_2 \lambda^2}{J^2} = g_s\,, \tag{6.1}$$

are both sufficiently small. Outside this regime, the continuum limit ceases to exist and the string worldsheet description breaks down. The region $g_2\lambda > J$ has indeed been identified in [1] as the regime where the strings start to expand into giant gravitons. It seems likely that a matrix (string) theory type description takes over in this regime. Indeed it would be interesting to derive or construct a dual $U(J)$ matrix model that reduces to our bit model in its strong coupling limit.

It should be straightforward to extend the string bit model to the quiver set-up of [18], which should in particular clarify the relation with matrix string theory. The central role of the permutation group further suggests that similar ideas as presented here may be used to construct an interacting string bit theory in the Penrose limit of AdS^3. Another remaining challenge is to construct the continuum limit of the bit theory. Some aspects of this problem, including (ways of avoiding) fermion doubling, have been analyzed in [32]–[34].

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